

## Solutions Linear Algebra II

1. Let  $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -4 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 3 & -6 & 9 \\ 0 & 2 & 3 \\ 0 & 0 & -2 \end{pmatrix}$ ,  $\mathbf{C} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$

Determine  $x = \frac{\det((\mathbf{A}')^{-1}) \cdot \det(\mathbf{A}^3 \cdot \mathbf{B} \cdot \mathbf{C}) \cdot \det(((\mathbf{A} + \mathbf{B})^{-1})^4)}{\det(\mathbf{C}^{-1})}$

We only need to calculate  $\det(\mathbf{A})$ ,  $\det(\mathbf{B})$ ,  $\det(\mathbf{C})$ , and  $\det(\mathbf{A} + \mathbf{B})$ , which all happen to be equal to  $-12$ . The rest can be done using rules for determinants.

$$\begin{aligned} x &= \frac{\det((\mathbf{A}')^{-1}) \cdot \det(\mathbf{A}^3 \cdot \mathbf{B} \cdot \mathbf{C}) \cdot \det(((\mathbf{A} + \mathbf{B})^{-1})^4)}{\det(\mathbf{C}^{-1})} \\ &= \frac{(\det(\mathbf{A}))^{-1} \cdot (\det(\mathbf{A}))^3 \cdot \det(\mathbf{B}) \cdot \det(\mathbf{C}) \cdot (\det(\mathbf{A} + \mathbf{B}))^{-4}}{(\det(\mathbf{C}))^{-1}} \\ &= -12 \end{aligned}$$

2. Calculate the determinant of  $\mathbf{A}_t = \begin{pmatrix} 1 & t & 0 \\ -2 & -2 & -1 \\ 0 & 1 & t \end{pmatrix}$ , and show that it is never 0.

$$\begin{aligned} \det(\mathbf{A}_t) &= 1(-2)t + t(-1)0 + 0(-2)1 - 0(-2)0 - 1(-1)1 - t(-2)t \\ &= 2t^2 - 2t + 1 \\ t_{1|2} &= \frac{2 \pm \sqrt{4 - 8}}{4} \end{aligned}$$

As the square root in the last equation is not defined, the determinant can never be zero.

3. Determine the inverse of the following matrices.

$$\mathbf{A} = (-\frac{1}{42}), \mathbf{B} = \begin{pmatrix} -1 & 8 \\ 2 & 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 4 & 3 \\ 0 & 2 & 0 \\ 2 & 4 & 7 \end{pmatrix}$$

$$\mathbf{A}^{-1} = (-42), \mathbf{B}^{-1} = -\frac{1}{20} \cdot \begin{pmatrix} 4 & -8 \\ -2 & -1 \end{pmatrix}, \mathbf{C}^{-1} = \begin{pmatrix} 7 & -8 & -3 \\ 0 & \frac{1}{2} & 0 \\ -2 & 2 & 1 \end{pmatrix}$$

We obtain the inverse of  $\mathbf{C}$  using the Gauss-Jordan algorithm.

$$\left( \begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 4 & 7 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} \text{I} - 2 \cdot \text{II} \\ \text{II} : 2 \\ \text{III} + 2 \cdot \text{II} \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 2 & 0 & 7 & 0 & -2 & 1 \end{array} \right) \quad \text{III} - 2 \cdot \text{I}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array} \right) \quad \text{I} - 3 \cdot \text{III}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -8 & -3 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array} \right)$$

4.  $\mathbf{D} = \begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 3 & -4 \\ 1 & 0 \end{pmatrix}, \mathbf{G} = \begin{pmatrix} 1 & 4 \\ -2 & -1 \end{pmatrix}$

Solve the matrix equation  $-\mathbf{D}^3 \cdot \mathbf{X} + \mathbf{D} \cdot \mathbf{F} = \mathbf{D}^2 \cdot \mathbf{I} \cdot \mathbf{D} \cdot \mathbf{X} \cdot \mathbf{I} - \mathbf{D} \cdot \mathbf{F} \cdot \mathbf{G}$ .

$$-\mathbf{D}^3 \cdot \mathbf{X} + \mathbf{D} \cdot \mathbf{F} = \mathbf{D}^2 \cdot \mathbf{I} \cdot \mathbf{D} \cdot \mathbf{X} \cdot \mathbf{I} - \mathbf{D} \cdot \mathbf{F} \cdot \mathbf{G}$$

$$-\mathbf{D}^3 \cdot \mathbf{X} + \mathbf{D} \cdot \mathbf{F} = \mathbf{D}^3 \cdot \mathbf{X} - \mathbf{D} \cdot \mathbf{F} \cdot \mathbf{G}$$

$$2\mathbf{D}^3 \mathbf{X} = \mathbf{D} \mathbf{F} (\mathbf{I} + \mathbf{G})$$

$$\mathbf{X} = \frac{1}{2} (\mathbf{D}^2)^{-1} \cdot \mathbf{F} (\mathbf{I} + \mathbf{G})$$

Now, we only need to calculate the result, which is  $\mathbf{X} = \begin{pmatrix} -1.25 & -4.5 \\ 1 & 2 \end{pmatrix}$ .

5.  $\mathbf{A} = \begin{pmatrix} 4 & -7 \\ 1 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & -7 \\ 1 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 & 2 \\ 2 & -1 \end{pmatrix}$

Solve the matrix equation  $3 \cdot \mathbf{X} - (\mathbf{A} - \mathbf{B})^3 \cdot \mathbf{X} - \mathbf{C} \cdot \mathbf{X} \cdot \mathbf{I} = \mathbf{I} - \mathbf{C}' \cdot \mathbf{X}$ .

$$3 \cdot \mathbf{X} - (\mathbf{A} - \mathbf{B})^3 \cdot \mathbf{X} - \mathbf{C} \cdot \mathbf{X} \cdot \mathbf{I} = \mathbf{I} - \mathbf{C}' \cdot \mathbf{X}$$

$$(3\mathbf{I} - \underbrace{(\mathbf{A} - \mathbf{B})^3}_{\mathbf{I}} \underbrace{-\mathbf{C} + \mathbf{C}'}_{=0}) \cdot \mathbf{X} = \mathbf{I}$$

$$\begin{aligned} 2\mathbf{IX} &= \mathbf{I} \\ \mathbf{X} &= \frac{1}{2}\mathbf{I} \end{aligned}$$

6. Consider the following system of equations.

$$\begin{aligned} x_1 - x_2 - x_3 &= 1 \\ -3x_1 + 4x_2 + 2x_3 &= -8 \\ -2x_1 + 3x_2 + x_3 &= -7 \end{aligned}$$

- (a) Rewrite the system of equations in matrix form.

$$\begin{pmatrix} 1 & -1 & -1 \\ -3 & 4 & 2 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \\ -7 \end{pmatrix}$$

- (b) How many solutions does the system have? Why?

Setting up the extended matrix yields:

$$\left( \begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ -3 & 4 & 2 & -8 \\ -2 & 3 & 1 & -7 \end{array} \right)$$

After a few steps of the Gauss Jordan algorithm we reach:

$$\left( \begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ -2 & 3 & 2 & -7 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

That shows that the matrix does not have full rank and includes the equation  $0x_1 + 0x_2 + 0x_3 = 0$ . Therefore, the system of equations has infinity many solutions.